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HAMBURGER-NOETHER MATRICES OVER RINGS

Julio CASTELLANOS
Departamento de Algebra, Facultad de Matemáticas, Universidad Complutense, Madrid, Spain

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We study Hamburger-Noether matrices over rings, obtaining some applications to deformations of curves and equisingularity. We also study a special type of them, the matrices of Arf.

Introduction

The Hamburger-Noether (H-N) matrix over a field has been introduced in [4,6] in relation with the process of resolution of the singularity of a branch. In this paper we study the H-N matrix over a ring that gives us for some rings a special parametrization for families of curves related with the process of the resolution of the singularities of the families. Campillo [2] defines the H-N expansion over a field and in [3] generalizes the H-N expansion over rings studying its relation with the equisingular deformation theory for plane curves.

In Section 1 we study the general properties, similarly to the curves case, i.e. multiplicity sequence, semigroup and conductor associated with the matrix. Along Section 2 we study the H-N matrix associated with some families of curves, given the monoidal transformation of a H-N matrix. We also study the relation of the existence of a H-N matrix for a family of branches with the equisingularity given by Zariski [11], and Stuz and Becker [9].

In Section 3 we define H-N-matrices of Arf and give a method to build them. They give us examples of flat deformations of a reduced curve.

1. Hamburger-Noether (H-N) matrices over a ring A

Along this section, A will be a commutative ring with unit. In [4,6] we have defined the Hamburger-Noether matrix, with entries over a field, associated to a twisted branch.

Definition 1.1. A *Hamburger-Noether* (H-N) *matrix over* A, is a matrix with entries in A with N rows, an infinite number of columns, composed by $r+1$ boxes C_i, of which the r-first boxes have a finite number of columns.

(i) Each box has a marked row which consists of $(1,0,\dots,0)$, the first entry after the box in the marked row is zero, and the first element different from zero in the marked row after the one is a unit in A. The matrix M has the form

$$M=\left(\begin{array}{cccc|cccc|c|ccc} 1 & 0 & \dots & 0 & 0 & & & & & & & \\ & & & & \dots & & & & & & & \\ & & & & & & & & & 1 & 0 & \dots \\ & & & & 1 & 0 & \dots & 0 & 0\ \dots & & & \\ & & C_0 & & & & C_1 & & & & C_r & \end{array}\right).$$

(ii) If the marked row i_k in the box C_g was marked last time in the box C_s, then all marked rows from C_s to C_g do not have a unit in the first column of the box C_g.

Associated with the matrix we have a Hamburger–Noether expansion as in [6]

$$\begin{aligned} &Y=A_{01}x_1+\cdots+A_{0h}x_1^h+Z_1x_1^h,\\ &\bar{Z}_0=A_{11}z_1+\cdots+A_{1h_1}z_1^{h_1}+Z_2z_1^{h_1},\\ &\dots\\ &\bar{Z}_{i-1}=A_{i1}z_i+\cdots+A_{ih_i}z_i^{h_i}+Z_{i+1}z_i^{h_i},\\ &\dots\\ &\bar{Z}_{r-1}=\sum_{1\le i}A_{ri}z_r^i \end{aligned}$$

such that

$$A_{ij}=\begin{pmatrix} a_{ij}^2\\ \vdots\\ a_{ij}^N\end{pmatrix},\qquad Z_i=\begin{pmatrix} z_{i2}\\ \vdots\\ z_{iN}\end{pmatrix},\qquad Y=\begin{pmatrix} x_2\\ \vdots\\ x_N\end{pmatrix}$$

and $\bar{Z}_{i-1}$ is obtained from Z_i in the following way: Let $z_i=z_{is}$; $z_{i-1}=z_{i-1m}$. $\bar{Z}_{i-1}$ is obtained from Z_i by taking z_i away and placing z_{i-1} between z_{im} and z_{im+1} if $m<s$, and between z_{im-1} and z_{im} if $s<m$. The matrices A_{i1} have a zero exactly in the entry where z_{i-1} is in $\bar{Z}_{i-1}$. The entries a_{ij}^k are obtained from the matrix M as follows:

If the box $C_0=(c_{0j}^k)$, $k=1,\dots,N$, $j=1,\dots,h$, with $c_{01}^1=1$, $c_{0j}^1=0$, $j>1$, $a_{0j}^k=c_{0j}^k$, $k>1$. $\bar{Z}_{i-1}$ corresponds to the box $C_i=(c_{ij}^k)$, $k=1,\dots,N$, $j=1,\dots,h_i$ in the following way:

Let $z_{i-1}=z_{i-1m}$ and m the index of the marked row in C_i;

(i) if $s\le m$, we fix $a_{ij}^{k+1}=c_{ij}^k$, $k<m$; $a_{ij}^k=c_{ij}^k$, $k>m$; and $z_i=z_{im}$.

(ii) if $m<s$, we fix $a_{ij}^{k+1}=c_{ij}^k$, $k<m-1$; $a_{ij}^k=c_{ij}^k$, $k>m-1$; and $z_i=z_{im+1}$.

Remark 1.2. If A is a domain, the matrix M considered with entries in the algebraic closure $\bar{K}$ of the quotient field K of A gives us an algebroid irreducible space curve

C. This curve C has the multiplicity sequence

$$\underbrace{n,\dots,n}_{h},\underbrace{n_1,\dots,n_1}_{h_1},\dots,n_r=1,$$

and the n_i's are given as follows:

Let $\alpha_{1k_0},\alpha_{i_1k_1},\dots,\alpha_{i_rk_r}$ be the elements of $M=(\alpha_{ij})$ different from zero after one of the marked rows; $1, i_1,\dots,i_r$ being the index of the marked rows, $h,h_1,\dots,h_{r-1}$ the length of the boxes. Then we set

$$\begin{aligned}
&n_r=1,\\
&n_{r-1}=k_{r-1}-(h+h_1+\cdots+h_{r-1}),\\
&\dots\\
&n_j=h_{j+1}n_{j+1}+\cdots+h_{j+s}n_{j+s}+(k_j-(h+h_1+\cdots+h_{j+s}))n_{j+s+1}\\
&\qquad\text{if } \alpha_{i_jk_j}\in C_{j+s+1},\\
&\dots\\
&n=h_1n_1+\cdots+h_sn_s+(k_0-(h+h_1+\cdots+h_s))n_{s+1}\quad \text{if } \alpha_{1k_0}\in C_{s+1},
\end{aligned}$$

In any case, for A we can associate with the matrix M the above numbers $E(M)=\{n,h,n_1,h_1,\dots,n_r=1\}$ and they only depend on the marked rows.

Remark 1.3. Each H–N matrix M gives us parametric equations $\Phi=\{\Phi_1(t),\dots,\Phi_N(t)\}\subset A[[t]]$, obtained making $z_r=t$ by successive substitutions on the H–N expansion associated with M. In the expansion, the expressions of $x_1,z_1,\dots,z_{r-1}$ as elements of $A[[t]]$ are $x_1=a_mt^m+a_{m+1}t^{m+1}+\cdots$, $z_i=b_{m_i}t^{m_i}+\cdots$, where a_n, b_{m_i} are units in A since, if the first entry after the one different from zero in the marked row of the box G_g is $c_{ik}^j\neq 0$ is a unit (Definition 1.1) and $z_g=z_{g+1}^{h_{g+1}}\cdot z_{g+2}^{h_{g+2}}\cdots z_i^{h_i}\cdot(c_{ik}^jz_{i+1}^k+\cdots)$ and, $x_1=x_1^hz_1^{h_1}\cdots z_i^{h_i}\cdot(c_{ik}^jz_{i+1}^k+\cdots)$ for similar case.

The above deformation verifies

Proposition 1.4. *A H–N matrix with entries in a domain A provides a morphism $\Phi:A[[\mathbf{X}]]\to A[[t]]$, $\mathbf{X}=(X_1,\dots,X_N)$ such that* Ker Φ *has height $N-1$.*

Proof. (i) We define $\Phi(X_i)=\Phi_i(t)$, $i=1,\dots,N$ as above. Let $\bar K$ be the algebraic closure of K, the quotient field of A and $\Phi':\bar K[[\mathbf{X}]]\to\bar K[[t]]$ the morphism induced by Φ. We have seen that Φ' gives us an algebroid irreducible curve over $\bar K$ (Remark 1.2), so (Ker Φ') has height $N-1$ and is prime.

(ii) The morphism $A\hookrightarrow\bar K$ is flat, so the exact sequence

$$0\longrightarrow \operatorname{Ker}\Phi\longrightarrow A[[\mathbf{X}]]\xrightarrow{\ \Phi\ } A[[t]]$$

gives the exact sequence

$$0\longrightarrow (\operatorname{Ker}\Phi)\otimes_A\bar K\longrightarrow A[[\mathbf{X}]]\otimes_A\bar K\xrightarrow{\ \Phi\otimes_A\ } A[[t]]\otimes_A\bar K.$$

We make the completion with respect to the ideal $(\mathbf{X})A[[\mathbf{X}]]=\mathfrak{N}$, i.e. with respect to the ideal $\mathfrak{N}\otimes_A\bar K$. We have $\Phi_1(t)=a_nt^n+a_{n+1}t^{n+1}+\cdots\in A[[t]]$ with a_n a unit in

A, and the morphism Φ is continuous for the topologies given by the ideals $\mathfrak{N}$ and $\mathfrak{N}_1=(t)A[[t]]$, since $(\mathfrak{N}_1)^n\subset\Phi(\mathfrak{N})$. Then to complete with $\Phi(\mathfrak{N})\otimes_A\bar{K}$ is the same as to complete with $\mathfrak{N}_1\otimes_A\bar{K}$ and we get $0\to(\operatorname{Ker}\Phi)\hat{\otimes}_A\bar{K}\to\bar{K}[[\mathbf{X}]]\xrightarrow{\Phi'}\bar{K}[[t]]$ and $(\operatorname{Ker}\Phi')\simeq(\operatorname{Ker}\Phi)\hat{\otimes}_A\bar{K}$ (1).

We have $A[[\mathbf{X}]]\otimes_A\bar{K}$ is A-flat, and completing with respect to the ideal $\mathfrak{N}\otimes_A\bar{K}$, we obtain that $\bar{K}[[\mathbf{X}]]$ is $A[[\mathbf{X}]]$-flat and $(\operatorname{Ker}\Phi)\hat{\otimes}_{A[[\mathbf{X}]]}\bar{K}[[\mathbf{X}]]\simeq(\operatorname{Ker}\Phi)\bar{K}[[\mathbf{X}]]$.

From $0\to(\operatorname{Ker}\Phi)\hat{\otimes}_A\bar{K}\to(\operatorname{Ker}\Phi)\hat{\otimes}_{A[[\mathbf{X}]]}\bar{K}[[\mathbf{X}]]$, and since $0\to(\operatorname{Ker}\Phi)\bar{K}[[\mathbf{X}]]\to(\operatorname{Ker}\Phi')$ we have $(\operatorname{Ker}\Phi)\bar{K}[[\mathbf{X}]]\simeq(\operatorname{Ker}\Phi')$.

(iii) From above, $(\operatorname{Ker}\Phi)\cap A=(0)$, and $\bar{K}[[\mathbf{X}]]_{(\operatorname{Ker}\Phi')}$ faithfully flat over $A[[\mathbf{X}]]_{(\operatorname{Ker}\Phi)}$ [8, 4.D] and we have $\operatorname{ht}(\operatorname{Ker}\Phi)=\operatorname{ht}(\operatorname{Ker}\Phi)\bar{K}[[\mathbf{X}]]$ [8, 13.B]. From (i), (ii) we obtain $\operatorname{ht}(\operatorname{Ker}\Phi)=N-1$. $\square$

Remark 1.5. Each H–N matrix M gives us a ring $R_M=A[[\Phi_1(t),\dots,\Phi_N(t)]]\subset A[[t]]$. We can consider the following semigroup $S_M\subset\mathbb{N}$:

Definition 1.6. Let M be a H–N matrix and $R_M\subset A[[t]]$ its ring associated as above, we set $S_M=\{\mathrm{o}(z(t))\mid z(t)\in R_M,\ z(t)=a_mt^m+a_{m+1}t^{m+1}+\cdots,a_m \text{ unit in } A\}$, where o is the order of the set power series.

The semigroup S of the branch associated with M (Remark 1.2) is $S_M\subset S$ but they can be different. The semigroup S_M corresponds to a semigroup of a branch:

Proposition 1.7. *The semigroup S_M defined above is a numerical semigroup, i.e.* $\#(\mathbb{N}-S_M)<\infty$.

Proof. Let n, $\beta_1=hn+n_1$, $\beta_2=hn+h_1\beta_1+n_2,\dots,\beta_r=hn+h_1\beta_1+\cdots+h_{r-1}\beta_{r-1}+1$. We shall prove that these elements belong to S_M.

(1) Let $x_1=a_nt^n+a_{n+1}t^{n+1}+\cdots$, a_n be a unit in A (Remark 1.3) and 1, i the indices of the marked rows in C_0, C_1. In the H–N expansion we have $x_i=a^i_{01}x_1+\cdots+a^i_{0h}x_1^h+x_1^hz_1$, so $x_1^hz_1=w_1\in R_M$, with $z_1=b_{m_1}t^{m_1}+\cdots+b_{m_1}$ a unit in A, and $\mathrm{o}(w_1)=hn+n_1=\beta_1\in S_M$.

(2) Let us suppose that for each $i<s$ there exists $w_i=x_1^hw_1^{h_1}\cdots w_{i-1}^{h_{i-1}}z_i\in R_n$ and $\mathrm{o}(w_i)=hn+h_1\beta_1+\cdots+h_{i-1}\beta_{i-1}+n_i=\beta_i\in S_M$. Let g be the index of the marked row in C_s and C_k the box before C_{s+1} having the same marked row. In the expansion we have:

(i) $z_k=a^g_{k+11}z_{k+1}+\cdots+a^g_{k+1h_{k+1}}z_{k+1}^{h_{k+1}}+z_{k+1}^{h_{k+1}}z^g_{k+2}$;

(ii) $z^g_{k+2}=a^{g'}_{k+31}z_{k+3}+\cdots+a^{g'}_{k+3h_{k+3}}z^{h_{k+3}}_{k+3}+z^{h_{k+3}}_{k+3}z^{g'}_{k+3}$, where $g'=g$ or $g+1$;

(iii) $z^j_{s-1}=a^j_{s-11}z_{s-1}+\cdots+a^j_{s-1h_{s-1}}z^{h_{s-1}}_{s-1}+z^{h_s}_{s-1}z_s$, where $j=g$ or $g+1$.

From (i), we obtain

$$\begin{aligned}&\frac{w_{k+1}}{z_{k+1}}\cdot\frac{w_k}{z_k}z_k-a^g_{k+11}w_{k+1}w_k\\&=w_{k+1}\frac{w_k}{z_k}(a^g_{k+12}z_{k+1}+\cdots+z^{h_k}_{k+1}z^g_{k+2})\in R_M;\end{aligned}$$

making the product with w_{k+1}/z_{k+1} and adding $-w_{k+1}^2(w_k/z_k)a_{k+12}^{g'}$ we get

$$w_{k+1}^2 \cdot \frac{w_k}{z_k} \cdot (a_{k+13}^g z_{k+1}^2 + \cdots + z_{k+1}^{h_k} z_{k+2}^g) \in R_M.$$

Continuing the process, in the step h_{k+1} we obtain

(iv) $w_{k+1}^{h_{k+1}} \cdot (w_k/z_k) z_{k+2}^g \in R_M$.

Now we substitute (iv) for (ii) and continue analogous to (i). We make the same process with the following boxes to (iii) and we get $(w_k/z_{h_s}) w_{k+1}^{h_{s+1}} \cdots w_{s-1}^{h_{s-1}} z_s \in R_M$, and have $w_k^{h_k} \in R_M$. Then $w_s = x_1^h w_1^{h_1} \cdots w_k^{h_k} \cdots w_{s-1}^{h_{s-1}} z_s \in R_M$ and $\mathrm{o}(w_s) = hn + h_1\beta_1 + \cdots + h_{s-1}\beta_{s-1} + n_s \in S_M$.

(3) Now g.c.d.$(n_1, \beta_1, \ldots, \beta_r) =$ g.c.d.$(n, n_1, \ldots, n_r = 1) = 1$ and so $\#(\mathbb{N} - S_M) < \infty$. $\square$

By the last proposition we can consider the conductor c_M of the semigroup, i.e. the element of it such that after it all elements of $\mathbb{N}$ belong to the semigroup. The conductor c_M is related with the conductor of $A[[t]]$ in R_M as follows:

Proposition 1.8. *Given R_M as above, $c_M \in \mathbb{N}$ the conductor of S_M, for all $w \in A[[t]]$ such that $\mathrm{o}(w) \geq c_M$ we have $w \in R_M$.*

Proof. Let $w = b_m t^m + b_{m+1} t^{m+1} + \cdots \in A[[t]]$, $\mathrm{o}(w) = m \geq c_M$, then there exist $z_m = a_m t^m + a_{m+1} t^{m+1} + \cdots \in R_M$, a_m a unit in A, such that $w = a_m^{-1} b_m z_m + w_1$, $w_1 \in A[[t]]$, $\mathrm{o}(w_1) > m$. After making the same process for w_1 and the next ones, since R_M is complete for the (t)-topology, we get $w - (a_m^{-1} b_m z_m + c_{m+1}^{-1} b_{m+1} z_{m+1} + \cdots) = 0$ and so $w \in R_M$. $\square$

In the case of a domain A, the conductor of the semigroup of the curve associated with the matrix (Remark 1.2) is less or equal than the conductor defined for the matrix.

From the above proposition we will see that only a few columns of the H–N matrix are important. Let us suppose that the marked rows in the matrix are numbered $1, \ldots, d$; we define for $i = 1, \ldots, d$, $m_i = (hn + h_1 n_1 + \cdots + h_{r-1} n_{r-1} + 1) - (h_{i_1} n_{i_1} + h_{i_2} n_{i_2} + \cdots + h_{i_k} n_{i_k})$ where the ith row has been marked in the boxes $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$. Let $\upsilon_i = h + h_1 + \cdots + h_{r-1} + c_M - m_i$ and $\upsilon = \max\{\upsilon_1, \ldots, \upsilon_d\}$. Then we have

Corollary 1.9. *If M is a H–N matrix having the same columns as M till the υth, then the matrices M, M' are equivalent in the sense that the two associated rings (Remark 1.5) R_M and $R_{M'}$ coincide.*

Remark 1.10. Given a H–N matrix M we can define invariants d_{ij} analogous to the ones we have defined in [4,6] for matrices of curves. These invariants are the invariants associated with the curve C obtained from M in the case of a domain A.

2. Equisingular deformation associated with a H-N matrix

Along this section we consider $A \simeq k[[V_1, \dots, V_l]]$. Let M be a H-N matrix over A, $R_M = A[[\Phi_1(t), \dots, \Phi_N(t)]] \subset A[[t]]$ its associated ring. Let $\mathfrak{m}_A$ be the maximal ideal of A, $\psi_i(t) = \operatorname{res} \Phi_i(t) = \Phi_i(t) + \mathfrak{m}_A A[[t]]$. The parametrization $\psi = \{\psi_1(t), \dots, \psi_N(t)\}$ is associated with the matrix $M_0 = (\bar{c}_{ij})$ obtained from $M = (c_{ij})$, where $\bar{c}_{ij} = c_{ij} + \mathfrak{m}_A$ are the residues $\operatorname{mod} \mathfrak{m}_A$ of the entries of M. By Definition 1.1(i) relating to some units in A, the parametrization ψ gives us an irreducible algebroid curve C_0 over k.

Definition 2.1. (Φ, Ψ, A) is a deformation of the parametrization ψ of the branch C_0 over A.

The morphism $\Phi : A[[\mathbf{X}]] \to A[[t]]$ given by $X_i = \Phi_i(t)$ (Proposition 1.4) provides us with a ring $R \simeq A[[\mathbf{X}]]/(\operatorname{Ker} \Phi)$ that can be identified with R_M over $A[[t]]$. We can consider (R, R_0, A, s) a deformation of the algebroid curve $R_0 \simeq R/\mathfrak{m}_A R$, having ψ as a parametrization, over A and with a section s given by the ideal $\mathfrak{p} = (X_1, \dots, X_n)R$.

Remark 2.2. The above deformation (R, R_0, A, s) does not have necessarily reduced the fiber in the origin, i.e. $R_0 \simeq R/\mathfrak{m}_A R$ can have nilpotents.

Example 2.3. $A = k[V]$,

$$M = \left(\begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & V & 0 & 0 & \dots \end{array}\right), \quad \Phi = \begin{cases} \Phi_1 = t^4 \\ \Phi_2 = t^5 \\ \Phi_3 = Vt^7 \end{cases} \quad \psi = \begin{cases} \psi_1 = t^4 \\ \psi_2 = t^5 \end{cases}$$

$R_0 = R[X_1 X_2 X_3]/(X_3, X_2^4 - X_1^5)$, but $X_3^2 - VX_2^2 X_1 \in (\operatorname{Ker} \Phi)$ and $X_3^2 \in (\operatorname{Ker} \Phi) + \mathfrak{m}_A A[[X_1, X_2, X_3]]$ and X_3 is nilpotent in $R/\mathfrak{m}_A R$.

We consider a generic curve of the deformation (R, R_0, A, s), $R_s = (\hat{R}_{\mathfrak{p}} \hat{\otimes}_{k(\mathfrak{p})} \overline{k(\mathfrak{p})})$ where $\overline{k(\mathfrak{p})}$ is the algebraic closure of $k(\mathfrak{p})$ and we have completed with respect to the maximal ideals in each case. If the characteristic of k is zero, R_s is reduced and does not depend on the coefficient field (a proof following E.G.A. 8, 11, 20, can be found in [5]). If the characteristic of k is different from zero, Abhyankar [1] gives examples where R_s is not reduced and also depends on the coefficient field.

In the case that R_s makes sense, R_s is the curve associated with the matrix M when A is domain (Remark 1.2).

Proposition 2.4. *The curves R_s and $(R_0)_{\text{red}}$ have the same multiplicity sequence, i.e. R is equisingular along the section s in the sense of Zariski* [11].

Proof. The first element of each marked row in the matrix M different from zero after the one is a unit in A (Definition 1.1). Then its residue mod $\mathfrak{m}_A$ is also different from zero, and the multiplicity sequences of R_s and $(R_0)_{\text{red}}$ coincide (Remark 1.2). □

Definition 2.5. Given a H-N matrix M over A, the monoidal transformation M_1 of M is the H-N matrix M_1 obtained from M by taking off the first column, and in the case that the first box of M has at least two columns, change the zero in the marked row on the second column of M by a marked one for M_1.

The above definition is compatible with the general meaning of monoidal transformation of an algebroid variety along a regular subvariety as follows:

Let (Φ, ψ, A) be a deformation of the parametrization associated with M, (R, R_0, A, s) the associated deformation and $\{x_1, \dots, x_n\}$ a basis for the ideal $\mathfrak{p}$ of s.

Proposition 2.6. *Let $R' = R[x_2/x_1, \dots, x_N/x_1]$, $R'_0 = R'/\mathfrak{m}_A R'$. Then (R', R'_0, A) is the deformation associated with (Φ', ψ', A) where $\Phi' = \{\Phi'_1(t) = \Phi_1(t), \Phi'_2(t) - a_{21}, \dots, \Phi'_N(t) - a_{N1}\}$, $\Phi'_i(t) = \Phi_i(t)/\Phi_1(t)$, $i \geq 2$, $a_{i1} \in A$, $\psi' = \{\psi'_1(t) = \psi_1(t), \psi'_2(t) - \bar{a}_{21}, \dots, \psi'_N(t) - \bar{a}_{N1}\}$, $\psi'_i(t) = \psi_N(t)/\psi_1(t)$, $i \geq 2$ is the quadratic transformation of the parametrization ψ.*

Proof. (i) $\Phi_1(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots$, a_n a unit in A (Remark 1.3), $\Phi_1(t) = u(t) \cdot t^n$, $u(t)$ a unit in $A[[t]]$. We have $\mathrm{o}(\Phi_i(t)) \geq n$, so $\Phi_i(t)/\Phi_1(t) = \Phi'_i(t) \in A[[t]]$ and

$$\psi'_i(t) = \frac{\psi_i(t)}{\psi_1(t)} = \frac{\mathrm{res}(\Phi_i(t))}{\mathrm{res}(\Phi_1(t))} = \mathrm{res}\left(\frac{\Phi_i(t)}{\Phi_1(t)}\right) = \mathrm{res}\, \Phi'_i(t).$$

To prove the rest we use the following lemma:

Lemma 2.7. *Let $z \in R$, $z \notin \mathfrak{m}_A R$, B a ring such that $A[[z]] \subset B \subset A[[t]]$. Then B is a $A[[z]]$-module of finite type, integer over $A[[z]]$, Noetherian, local, complete, $\dim B = \dim A + 1$ and it has k as coefficient field.*

By Lemma 2.7, $R \simeq A[[x_1]][x_2, \dots, x_N]$, and $R[x_2/x_1, \dots, x_N/x_1]$ is integer over $A[[x_1]]$, local, complete and with the same dimension as R, i.e. $s+1$. Let $\Phi' : A[[\mathbf{X}']] \to A[[t]]$ be defined by $\Phi'(X'_i) = \Phi'_i(t)$ and $R' = A[[\mathbf{X}']]/(\mathrm{Ker}\, \Phi')$, and $x'_i = X'_i + (\mathrm{Ker}\, \Phi')$, we can identify over $A[[t]]$ x_1 with x'_1 and x'_i with x_i/x_1. By the lemma, $R' \simeq A[[x'_1]][x'_2, \dots, x'_N]$ and $R[x_2/x_1, \dots, x_N/x_1] \simeq A[[x_1]][x_2, \dots, x_n, x_2/x_1, \dots, x_N/x_1] = A[[x_1]][x_2/x_1, \dots, x_N/x_1] \simeq R'$.

From above we get that $\Phi'_i = \Phi_i/\Phi_1 = a_{i1} + t^{m_i} \cdot a(t)$, $a_{i1} \in A$, $i = 2, \dots, N$ where $(1, a_{21}, \dots, a_{N1})$ is the first column of M and the H-N matrix associated with $\{\Phi'_1, \Phi'_2 - a_{21}, \dots, \Phi'_N - a_{N1}\}$ is M' obtained from M by taking off the first column. □

Let us see now that the monoidal transformation of R with center the ideal $\mathfrak{p}$ has exactly one strict transform which corresponds to R'.

We follow Zariski for the case of curves [12]. We set $\mathrm{Bl}_{\mathfrak{p}}(R)=\bigcup_{i=1}^{N}(\operatorname{Spec} R_{x_i})$, $R_{x_i}=R[x_1/x_i,\dots,x_N/x_i]$, and define an equivalence relation $\sim$ for all $\Omega_{x_i}\in\operatorname{Spec}(R_{x_i})$, $\Omega_{x_j}\in\operatorname{Spec}(R_j)$, $\Omega_{x_i}\sim\Omega_{x_j}$ if and only if $(R_{x_i})_{\Omega_{x_i}}\simeq(R_{x_j})_{\Omega_{x_j}}$.

Given x_i, x_j, we set R_{x_i,x_j} the localization of R_{x_i} by the powers $(x_i/x_j)^m$, $m\geq 0$. Then we have

Lemma 2.8. (i) $R_{x_i,x_j}=R_{x_i}[x_i/x_j]$.
(ii) $R_{x_i,x_j}=R_{x_j,x_i}$.

Proposition 2.9. *Let R and $x_1,\dots,x_n$ be as in the hypothesis of Proposition* 2.6. *Then*

(i) *x_i/x_1 is a unit in R_{x_i}, for all $i=2,\dots,N$;*
(ii) *$R_{x_i}=R_{x_1,x_i}$ for all $i=2,\dots,N$;*
(iii) *There exists a bijection from* $\operatorname{Spec}(R_1)$ *onto* $T=\mathrm{Bl}_{\mathfrak{p}}(R)/\sim$.

Remark 2.10. From the above proposition we have that all monoidal transforms of R with center $\mathfrak{p}$ are obtained by localizations of $R[x_2/x_1,\dots,x_N/x_1]$ in their maximal ideal and completation with respect to those ideals. But by Lemma 2.7 that ring is local and complete, and so $R[x_2/x_1,\dots,x_N/x_1]$ is the unique monoidal transform of R with center $\mathfrak{p}$.

Stuz and Becker [9], in the analytic case, have made a generalization of the equisingularity of Zariski for hypersurfaces, to the general case.

In the formal case and for irreducible varieties, i.e. R and $R_{\mathfrak{p}}$ domains, we have:

Definition 2.11. Let R be the ring of an algebroid irreducible variety, $\mathfrak{p}$ an ideal of R with $R/\mathfrak{p}$ regular, the ring of the subvariety of the singular points, and $R_{\mathfrak{p}}$ domain R is *equisingular along* $\mathfrak{p}$ [9] if

(i) $\tilde{\Pi}_1:\mathrm{Bl}_{\mathfrak{p}}(R)\to\operatorname{Spec}(R)$ is finite and for all closed points $\mathfrak{m}_1\in\tilde{\Pi}_1^{-1}(\mathfrak{m})$ the monoidal transform in $\mathfrak{m}_1$, $R_1=\operatorname{Spec}(\mathrm{Bl}_{\mathfrak{p}}(R)_{\mathfrak{m}_1})\hat{}$, $\Pi_1:\operatorname{Spec}(R_1)\to\operatorname{Spec}(R)$ does not depend on the chosen point $\mathfrak{m}_1$;

(ii) Let $\mathfrak{p}_{i-1}\in\operatorname{Spec} R_{i-1}$ be a minimal primary lying over $\mathfrak{p}_{i-2}$ by the morphism Π_{i-1}. Then the morphism $\tilde{\Pi}_i:\mathrm{Bl}_{\mathfrak{p}_{i-1}}(R_{i-1})\to\operatorname{Spec}(R_{i-1})$ is finite, and if $\mathfrak{m}_{i-1}$ is the maximal ideal of R_{i-1}, the morphism $\Pi_i:\operatorname{Spec}(R_i)=\operatorname{Spec}(\mathrm{Bl}_{\mathfrak{p}_{i-1}}(R_{i-1})_{\mathfrak{m}_{i-1}})\hat{}\to\operatorname{Spec}(R_{i-1})$ does not depend on the chosen closed point $\mathfrak{m}_i\in\tilde{\Pi}_{i-1}^{-1}(\mathfrak{m}_{i-1})$;

(iii) For all i, either R_i is regular or its singular locus is $\Pi_i^{-1}\cdots\Pi_1^{-1}(V(\mathfrak{p}))$;

(iv) There exist $s\in\mathbb{N}$ such that R_s is regular (the R_j is regular for $j>s$) and $\mathfrak{p}_s^*={}^*\Pi_s^{-1}\cdots{}^*\Pi_1^{-1}(\mathfrak{p})$ (${}^*\Pi_i^{-1}:R_{i-1}\hookrightarrow R_i$ the associated morphism to Π_i) satisfies $R_s/\sqrt{\mathfrak{p}_s^*}\simeq R/\mathfrak{p}$.

This definition is given for characteristic zero and it is independent of the morphism $\operatorname{Spec} R\to\operatorname{Spec} A$ that makes R a deformation of a curve R_0 over A.

Proposition 2.12. *The deformation (R, R_0, A) associated with a H–N matrix M verifies that R is equisingular along $\mathfrak{p}$, ideal of R given by the section determined by M.*

Proof. Let $\mathfrak{p}=(x_1,\dots,x_N)R$, $x_i=X_i+(\operatorname{Ker}\Phi)$. We have seen that, in this case, there is only one monoidal transform $R[x_2/x_1,\dots,x_N/x_1]$ (Remark 2.10) and the extension $R\hookrightarrow R[x_2/x_1,\dots,x_N/x_1]$ is finite (proof of Proposition 2.6). Then $\tilde{\Pi}_1|_{R_{x_1}}:\operatorname{Spec}(R_{x_1})\to \operatorname{Spec}(R)$ is finite, by Proposition 2.9, $\tilde{\Pi}_j|_{R_{x_j}}:\operatorname{Spec}(R_{x_j})\to\operatorname{Spec}(R)$ is finite, and $\tilde{\Pi}_1:\operatorname{Bl}_{\mathfrak{p}}(R)\to\operatorname{Spec}(R)$ is finite.

Let $\mathfrak{p}_1=(x_1', x_2'-a_{21},\dots,x_N'-a_{N1})$, $(1,a_{21},\dots,a_{N1})$ the first column of M, the ideal of R_1 lying over $\mathfrak{p}$, and M_1 the matrix obtained from M taking off the first column. M_1, R_1, $\mathfrak{p}_1$ satisfy the same conditions as $R, M, \mathfrak{p}$. Then inductively we obtain $M_i, R_i, \mathfrak{p}_i$. If $s=h+h_1+\cdots+h_{s-1}$, the matrix M_s corresponds to the last box C_s of M, the parametrization associated is $\Phi_s=\{\Phi_1^s(t),\dots,\Phi_N^s(t)\}$ and there exists i such that $\Phi_i^s=t$ and $R_s\simeq A[[t]]$ is regular.

For $i<s$, R_i has a matrix M_i with multiplicity sequence different from 1. Then $\mathfrak{p}_i=(x_1^i-a_{1i},\dots,x_N^i-a_{Ni})R_i$ is singular because the generic curve along $\mathfrak{p}_i$ has the multiplicity sequence of the matrix M_i (Remark 1.2).

Let $R_s\simeq A[[t]]$. Then $*\Pi_s^{-1}\cdots *\Pi_1^{-1}(\mathfrak{p})=\mathfrak{p}R_s\simeq\mathfrak{p}A[t]\simeq x_1A[[t]]$, where $x_1=a_nt^n+a_{n+1}t^{n+1}+\cdots$, a_n a unit in A. Then $R_s/\sqrt{\mathfrak{p}R_s}\simeq A$. □

Remark 2.13. The deformation (R, R_0, A, s) associated to a H–N matrix has s as the unique singular section since, from Propositions 2.9 and 2.12, by blowing up successively the special section one gets the regular scheme $\operatorname{Spec} A[[t]]$.

In the case that we consider R a domain of an algebroid variety, $\dim R=l+1$, $\mathfrak{p}$ the ideal of the singular locus of R, with $\dim R/\mathfrak{p}=l$, if we have also a morphism $A\hookrightarrow R$ that makes R a deformation of $R_0\simeq R/\mathfrak{m}_AR$ domain, over A, i.e. the curve $\operatorname{Spec}(R_0)$ and the variety $\operatorname{Spec}(R)$ are irreducible. We have

Proposition 2.14. *Let (R, R_0, A, s) as above equisingular along a section s of ideal $\mathfrak{p}$ of R (Stuz). Then there exists a H–N matrix M over A such that its deformation associated is (R, R_0, A, S).*

Proof. We have to build a H–N matrix M for R.

(i) Let us see that R has a parametrization in $A[[t]]$ for the section given by $\mathfrak{p}$.

Let $\mathfrak{m}$ be the maximal ideal of R, and $\mathfrak{p}=(x_1,\dots,x_N)R$, $\Pi_1:\operatorname{Bl}_{\mathfrak{p}}(R)\to\operatorname{Spec}(R)$ is finite and $\Pi^{-1}(\mathfrak{m})=\mathfrak{m}_1$ because otherwise R_0 would be reducible. Let $x_1\in R$ be such that $\mathfrak{m}_1\in\operatorname{Spec}(R[x_2/x_1,\dots,x_N/x_1])$, $R\hookrightarrow R[x_2/x_1,\dots,x_N/x_1]$ is finite, and it contains only one maximal ideal $\mathfrak{m}_1$, we get that $R[x_2/x_1,\dots,x_N/x_1]$ is local complete and $R_1=(\operatorname{Bl}_{\mathfrak{p}}(R)_{\mathfrak{m}_1})^{\wedge}\simeq R[x_2/x_1,\dots,x_N/x_1]$. By the induction of Definition 2.11, if $\mathfrak{p}_1=(x_1',\dots,x_N')\subset R_1$ lying over $\mathfrak{p}$, then we get $R_2\simeq R_1[x_1'/x_i,\dots,x_N'/x_i]$. Then we obtain a chain $R \subset R_1 \subset \cdots \subset R_s$ where R_i is local, complete and finite over R_{i-1}, and R_s is regular. Let us show that $R_s \simeq A[[t]]$.

We have $A = K[[V_1, \dots, V_l]]$, $R \simeq A[[\mathbf{X}]]/I$, $I \cap A = (0)$, $R/\mathfrak{m}_A R \simeq R_0$ with dimension one, and $\{V_1, \dots, V_l, x_1\}$ is a parameter system of R. Then $A[[x_1]] = K[[V_1, \dots, V_l, x_1]] \subset R \subset R_s$ is finite, $\dim R_s = l+1$ and $R_s \simeq A[[t]]$. So we have a parametrization $x_i \to \Phi_i(t) \in A[[t]]$ compatible with R. We have $\Phi_1(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots$, with a_n a unit in A, since by hypothesis $A[[t]]/\sqrt{x_1 A[[t]]} \simeq R/\mathfrak{p} \simeq A$, so there exists $u_1(t) = b_0 + b_1 t + \cdots \in A[[t]]$ with $\Phi_1(t) \cdot u_1(t) = t^n$, and a_n, b_0 are units in A.

(ii) Now we build a H–N matrix for R with the parametrization Φ. We have $R_1 \hookrightarrow A[[t]]$, the image of x_i/x_1 is $a_{i1} + b_i t^{r_i} + \cdots$, $a_{i1} \in A$, $i = 2, \dots, N$. We consider the ideal $\mathfrak{p}_1 = (x_1, x_2', \dots, x_N')$, $x_i' = x_i/x_1 - a_{i1}$, $\mathfrak{p}_1$ is prime and $\mathfrak{p}_1 \cap R = \mathfrak{p}$. Then we have

$$\left\{\begin{array}{l} x_2 = a_{21} x_1 + x_2' x_1 \\ \cdots \\ x_N = a_{N1} x_1 + x_N' x_1 \end{array}\right. \quad \to \quad \begin{pmatrix} 1 \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix}$$

is the first column of the H–N matrix for R.

Doing the same for R_1 and $\mathfrak{p}_1$, we can distinguish the following two cases:

(a) $R_2 = R_1[x_2'/x_1, \dots, x_N'/x_1]$; then we set $\mathfrak{p}_2 = (x_1, x_2'', \dots, x_N'')$, $x_i'' = x_i'/x_1 - a_{i2}$ where the image of x_i'/x_1 in $A[[t]]$ is $a_{i2} + b_i t^{s_i} + \cdots$, $a_{i2} \in A$ and

$$\left\{\begin{array}{l} x_2 = a_{21} x_1 + a_{22} x_1^2 + x_2'' x_1 \\ \cdots \\ x_N = a_{N1} x_1 + a_{N2} x_1^2 + x_N'' x_1 \end{array}\right. \quad \to \quad \begin{pmatrix} 0 \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix}$$

is the second column of the matrix.

(b) $R_2 = R_1[x_1/x_j', \dots, x_N'/x_j']$; then we set $\mathfrak{p}_2 = (x_1'', x_2'', \dots, x_j', \dots, x_N'') R_2$, $x_1'' = x_1/x_j'$, $x_i'' = x_i'/x_j' - a_{i2}$, $i \neq 1$ and $a_{i2} + b_i t^{s_i} + \cdots$ the image of x_i'/x_j in $A[[t]]$. The second column of the matrix is $(0, a_{22}, \dots, 1^{(j)}, \dots, a_{N2})$.

Continuing the process we get $R_s = R_{s-1}[y_1/y_k, \dots, y_N/y_k] \simeq A[[t]]$, with $y_k = b_1 t + b_2 t^2 + \cdots$, and from R_s we can always divide by y_k and we obtain the last box of the matrix.

We have to verify now that the first entry different from zero after the one in each marked row is a unit. That entry is associated with $x_1, z_1, \dots, z_s$ in the H–N expansion, and their lower coefficients are units in A since the lower coefficients of x_1 is a unit (i). Then working analogously to Remark 1.3, but in the opposite way, we get the result.

To finish the proof, we have that the numbers n, h, $n_1, h_1, \dots, n_r = 1$ associated with M (Remark 1.2) correspond to a multiplicity sequence of the branch R_0 or R_s if we consider the entries of the matrix mod $\mathfrak{m}_A$, or in q.f. (A). $\square$

3. H–N matrices of Arf over A

Along this section, $A=K[[V_1,\dots,V_l]]$. In [7] Lipman defines an Arf curve as Spec R, R a one-dimensional equicharacteristic domain over an algebraically closed field K and its semigroup of values in its integral closure $\bar{R}\simeq K[[t]]$ is $\Gamma^*=\{n,2n,\dots,hn+n_1,\dots,hn+h_1n_1+\cdots n_r+\mathbb{N}\}$ where the multiplicity sequence of R is

$$\Big\{\underbrace{n,\dots,n}_{h},\underbrace{n_1,\dots,n_1}_{h_1},\dots,n_r=1,\Big\}$$

Definition 3.1. A H–N matrix M is of Arf if its semigroup (Definition 1.6) is $S_M=\{n,2n,\dots,hn+n_1,\dots,hn+h_1n_1+\cdots,n_2\mathbb{N}\}$ and it has n rows, where $E(M)=n,h,n_1,h_1,\dots,n_r=1$ are the numbers associated with the matrix (Remark 1.2).

Definition 3.2. Given a H–N matrix M, the Arf closure of M is a H–M matrix of Arf M^* such that its associated ring R^* is $R\subset R^*\subset A[[t]]$, and R^* is the smallest among the rings between R and $A[[t]]$ having the numbers associated with M^* (Remark 1.2) $E(M^*)=E(M)$.

We can build the Arf closure of a matrix M, doing a process similar to the case of curves [6].

(i) We consider the matrix M_d formed with the d marked rows.

(ii) Let B^* be the Apery basis of the semigroup Γ^*, i.e.

$$B^*=\{\beta_0=n,\ \beta_i=\min\{\gamma\in\Gamma^*\colon \gamma_i\equiv i(n),\ i=1,\dots,n-1\}\}$$

Lemma 3.3. *Let M be a H–N matrix over A, $i_1,\dots,i_d$ the indices of the different marked rows in M and $C_{j_1},\dots,C_{j_d}$ the boxes where the above rows have been marked for the first time. Then $\beta_k=hn+h_1n_1+\cdots+n_{j_k}$, $k=1,\dots,d$ belong to the Apery basis B^* of the semigroup Γ^*.*

Proof. The proof is similar to the one given in [6]. Let $\alpha_0,\dots,\alpha_s\in B^*$ be such that $\alpha_i<\beta_k$; and let M_d be the H–N matrix corresponding to the marked rows of M. Then we have $E(M_d)=E(M)$ (Remark 1.2). We build a H–N matrix M' by adding to M_d for each α_i, a row with a 1 in the column c_{ik_i} if $\alpha_i=hn+h_1n_1+\cdots+k_in_i$, and zeros in the rest. M' is such that $E(M')=E(M_d)$. Suppose $\beta_k\notin B^*$, i.e. β_k belongs to the semigroup generated by $\alpha_0,\dots,\alpha_s$. Let $\Phi'=\{\Phi_{i_1}(t),\dots,\Phi_{i_d}(t),\eta_0(t),\dots,\eta_s(t)\}\subset A[[t]]$ be the parametrization associated with M', as $\beta_k=hn+h_1n_1+\cdots+n_{j_k}$ and it corresponds in the parametrization to $\Phi_{i_k}(t)=a_{r1}t^{r_1}+\cdots$, since $r_1\in\Gamma^*$, from above there exists $P_1(\eta_0,\dots,\eta_s)\in A[[\eta_0,\dots,\eta_s]]$ such that $r_1<\mathrm{o}(\Phi_{i_k}-P_1(\eta_0,\dots,\eta_s))\in\Gamma^*$. Repeating the process we get $P(\eta_0,\dots,\eta_s)\in A[[(\eta_0,\dots,\eta_s)]]$ with $\mathrm{o}(\Phi_{i_k}-P(\eta_0,\dots,\eta_s))>\beta_k$, and the parametrization $\eta=\{\Phi_{i_1},\dots,\Phi_{i_k}-P(\eta_0,\dots,\eta_s),\dots,\Phi_{i_d},\eta_0,\dots,\eta_s\}$ gives us also the ring R' associated with M' and Φ'. Now we can consider the residue by $\mathfrak{m}_A$ for the two parametrizations Φ', η, if we denote $\bar{\Phi}'_{i_j} = \mathrm{res}(\Phi'_{i_j})$, $\bar{\eta}_i = \mathrm{res}(\eta_i)$, $\bar{\eta} = \{\bar{\Phi}'_{i_1}, \dots, \bar{\Phi}'_{i_k} - \bar{P}(\eta_0, \dots, \eta_s), \dots, \bar{\Phi}_{i_d}, \bar{\eta}_0, \dots, \bar{\eta}_s\}$, if $P(\eta_0, \dots, \eta_s) = P_1 + \cdots + P_t$ where the P_i's are monomial, its coefficients are a_{r_j}, coefficients of $\Phi_{i_k}(t)$ and are units in A, because $r_j \in \Gamma^*$. Then we have a matrix $\bar{M}_1$ associated with $\bar{\eta}$, such that it has no 1 in the entry $c^{i_k}_{ij_k}$, so $E(\bar{M}_1) \neq E(\bar{M}')$, but the parametrizations $\bar{\Phi}'$, $\bar{\eta}$ correspond to the same branch over k, and $E(\bar{M})$, $E(\bar{M}')$ are the multiplicity sequence of the branch, which is an invariant of it.

Then β_k does not belong to the semigroup generated by $\alpha_0, \dots, \alpha_s$ and so $\beta_k \in B^*$. □

Proposition 3.4. *The closure M^* of the H–N matrix M is obtained from M as follows. Its d first rows are the marked rows $\{i_1, \dots, i_d\}$ in M; and we add for each $\alpha_i \in B^* - \{\beta_1, \dots, \beta_d\}$, with β_i as in Lemma 3.3 and $\alpha_i = hn + h_1 n_1 + \cdots + k_i n_i$ a row with a 1 in the column c_{ik_i} and zeros in the rest.*

Proof. The matrices M and M^* have the marked rows in common so $E(M) = E(M^*)$ (Remark 1.2).

The semigroup of values S_{M^*} contains $B^* - \{\beta_1, \dots, \beta_d\}$ by construction. Suppose that β_k is the minimum value that does not belong to S_{M^*}; we add to the matrix M^* $(s+1)$ rows, similarly to Lemma 3.3, associated with $\{\alpha_0, \dots, \alpha_s\} \subset S_{M^*}$ and $\alpha_i < \beta_k$. Then we get a contradiction with Lemma 3.3 and hence $S_{M^*} = \Gamma^*$.

Let $R^* \subset A[[t]]$ be the ring associated with M^*. Let us show now that $R \subset R^*$. $S_M \subset \Gamma^*$ since the semigroup S of the branch associated with M is $S_M \subset S$ (Definition 1.6) and in general, is $S \subset \Gamma^*$. Then if $x \in R$, there exists $z_1 \in R^*$ with $\mathrm{o}(z_1) < \mathrm{o}(x - z_1) = \alpha_1$. So $\alpha_1 \in \Gamma^*$, since on the opposite $\alpha_1 = hn + h_1 n_1 + \cdots + k_i n_i + m$ with the H–N matrix associated with the parametrization of $R^*\{\Phi_{i_1}, \dots, \Phi_{i_d}, \eta_{d+1}, \dots, \eta_N, x - z_1\}$ has marked row in C_{i+1} different from the marked row in C_{i+1} in M^*. Repeating the process, we get $z_i \in R^*$ for $i \geq 1$ such that there is a $k_0 \in \Gamma^*$ for each $k > k_0$, $k_0 < \mathrm{o}(x - \sum_{i=1}^{k} z_i) \in \Gamma^*$, and as R^* is complete $x \in R^*$.

By construction, M^* has n rows, and finally M^* is the Arf closure of M. □

The matrices of Arf have a good behaviour from the point of view of the theory of deformations.

Proposition 3.5. *Given a H–N matrix M^* of Arf, we have:*

(i) *Its associated deformation (R^*, R^*_0, A, s) is a flat deformation over A with the fiber at the origin R^*_0 reduced.*

(ii) *The curves R^*_s and R^*_0 are Arf.*

(iii) *The successive monoidal transforms of R^* are of Arf and they satisfy the same results.*

Proof. (i) The deformation Φ of the parametrization associated with M^* has the semigroup constant, i.e., the parametrizations of the branch R_0^*, $\operatorname{res}(\Phi) \bmod \mathfrak{m}_A$, and $\Phi_\mathfrak{p}'$ of R_s^* obtained by considering Φ in $\overline{k(\mathfrak{p})}$ (algebraic closure of $k(\mathfrak{p})$, residual field of $\mathfrak{p}$, ideal of s) have the same semigroup of values. Then the deformation is flat with R_0^* reduced as follows from $\delta(R_s^*)=\delta((R_0^*)_{\mathrm{red}})$ and the proof of [10, Proposition (1), 3.3].

(ii) Let R_s^* and R_0^* be curves having the matrices associated, M_0 with the entries of M after res of $\mathfrak{m}_A$, and M_s obtained by considering the entries of M in $\overline{k(\mathfrak{p})}$ (as above). They satisfy $\Gamma^*=S_{M^*}\subset S(R_s^*)$, $\Gamma^*\subset S(R_0^*)$, and $E(R_0^*)=E(R_s^*)=E(M^*)$ so the semigroups must coincide, $S(R_s^*)=S(R_0^*)=\Gamma^*$, and the two curves are of Arf [7].

The monoidal transform of R along s is the ring R_i associated with the matrix M_1 obtained by taking off the first column of M (Proposition 2.6). Then M_1 is a matrix having $E(M_1)=\{n,h-1,n_1,h_1,\ldots,n_2=1\}$ and the semigroup $S_{M_1}=\{n,2n,\ldots,(h-1)n,(h-1)n+n_1,\ldots,(h-1)n+h_1n_1+\cdots+n_r+\mathbb{N}\}$, as can easily be shown by looking at the expression of the monoidal transform in function of the parametrization (Proposition 2.6). Hence the monoidal transform M_1^* is of Arf and satisfies (i) and (ii). □

Corollary 3.6. *The deformation associated (R^*,R_0,A,s) with a H-N matrix of Arf M^* is equisingular in the senses of Zariski* [11] *and Stuz and Becker* [9]. □

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